Chaotic dynamics and reversal statistics of the forced spherical pendulum: comparing the Miles equations with experiment

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The forced spherical pendulum is of intrinsic interest to dynamicists as well as to geophysicists as a simple mechanical analogue of the polarity reversals of the Earth’s magnetic field. The system displays chaotic dynamics involving irregular reversals of its direction of motion, both in terms of its angular momentum and of its direction of precession. Here, we analyse the difference between angular-momentum and precession reversals and compare the results of experimental work that has been performed on chaotic reversals in a laboratory pendulum with numerical simulations of the Miles equations that represent the pendulum dynamics; we find good agreement.

Keywords: spherical pendulum; chaos; Miles equations

AMS Subject Classifications: 34C15; 34C28; 37D45; 70K40; 86A25

1. Introduction

The motion of a free, undamped spherical pendulum is a classical problem (see, e.g. [1]), to which the solution is well known. For small-amplitude oscillations, the projection of the pendulum bob on a horizontal plane is in general a quasi-ellipse: a figure like an ellipse, but not quite closed, of constant shape but slowly precessing axes. The precession of the quasi-ellipse is in the direction of rotation of the bob – that is to say, of the angular momentum of the system – and occurs at a constant rate [2–4]. Two special cases occur when the quasi-ellipse is a circle, which is known as a conical pendulum, and when it is a line; the system then being a simple pendulum. There is no precession in either of these cases. (Hence, the constructor of a Foucault pendulum wishing to measure the precession arising from the Earth’s rotation must be careful to ensure that the system is a simple and not a spherical pendulum to avoid spurious results from this source [5]. Of course, the reverse is also true: the Coriolis force from the Earth’s rotation acts upon a forced spherical pendulum experiment in the laboratory. It should introduce a very slow modulation of the dynamics, which was undetectable in our experiments.)

In 1962, Miles published a paper discussing the response of a lightly damped spherical pendulum to small amplitude, simple harmonic forcing of its point of suspension [6]. He found that the system is governed by a set of four first-order, autonomous nonlinear
 ordinary differential equations, hereinafter called the Miles equations. Further analysis led
him to discover that, for sufficiently light damping, planar simple harmonic motion – the
intuitively obvious result of the forcing – is unstable for forcing in the neighbourhood of
the natural frequency of the pendulum. Furthermore, non-planar simple harmonic motion
is stable in a forcing frequency range overlapping the regions of stable and unstable planar
motion, and, most interesting of all, there is a finite forcing frequency range near the
natural frequency of the pendulum for which no stable simple harmonic motion is
possible, either planar or non-planar.

At that time, as Miles later recalled [7], he did not fully realize the significance of this
finding, and it was left to Lorenz to discover a short while later that the solutions of a set
of first-order, autonomous, nonlinear ordinary differential equations can be chaotic [8],
and thus to be the first to appreciate that what is now known as deterministic chaos can
exist in dissipative systems. The Miles equations are, like the Lorenz equations, dissipative.
On one hand, they are more complex in comprising four equations rather than three; but
on the other they are simpler in having only two controlling parameters rather than the
three of the Lorenz equations.

2. The Miles equations
Miles [6,7] considered a simple pendulum of mass \( m \) and length \( l \), having a small-amplitude
natural frequency \( \omega_0 = \sqrt{g/l} \). The point of suspension of the pendulum, with
Cartesian coordinates \((x_0, y_0)\), is oscillated horizontally in a straight line, so that
\((x_0, y_0) = (\varepsilon l \cos \omega t, 0)\), where \( 0 < \varepsilon \ll l \); that is with small oscillations, \( \varepsilon \) being the ratio
of the amplitude of the oscillations to the length of the pendulum.

Miles sought a solution for the horizontal displacement of the pendulum bob of
the form

\[
\begin{align*}
x &= \varepsilon^{1/3} [p_1(\tau) \cos \omega t + q_1(\tau) \sin \omega t], \\
y &= \varepsilon^{1/3} [p_2(\tau) \cos \omega t + q_2(\tau) \sin \omega t],
\end{align*}
\]

where \( \tau = \varepsilon^{2/3} \omega t / 2 \) is a dimensionless slow time and \( p_1, q_1, p_2, q_2 \), are the slowly varying
coordinates of the pendulum in a four-dimensional phase space. By analysing
the equations of motion and discarding terms above third order, Miles obtained the
following system of equations:

\[
\begin{align*}
\dot{p}_1 &= -a p_1 - \left( v + \frac{E}{8} \right) q_1 - \frac{3M}{4} p_2, \\
\dot{q}_1 &= -a q_1 + \left( v + \frac{E}{8} \right) p_1 - \frac{3M}{4} q_2 + 1, \\
\dot{p}_2 &= -a p_2 - \left( v + \frac{E}{8} \right) q_2 + \frac{3M}{4} p_1, \\
\dot{q}_2 &= -a q_2 + \left( v + \frac{E}{8} \right) p_2 + \frac{3M}{4} q_1,
\end{align*}
\]

which we term the Miles equations, where the dots imply differentiation with respect to \( \tau \); 
\( E = p_1^2 + p_2^2 + q_1^2 + q_2^2 \) is an energy and \( M = p_1 q_2 - p_2 q_1 \) is an angular momentum.
\( \alpha \) and \( \nu \) are the damping and tuning parameters of the pendulum:

\[
\alpha = \frac{2\delta \omega_0}{\varepsilon^2/\beta \omega},
\]

where \( \delta \) is the damping ratio – \( \delta = 1 \) is critical damping – and

\[
\nu = \frac{\omega^2 - \omega_0^2}{\varepsilon^2/\beta \omega^2},
\]

measures the frequency shift from resonance.

Note that the Miles equations are invariant under the transformation \((p_2, q_2) \rightarrow (-p_2, -q_2)\), as the system is symmetric about the \( y \)-axis. The equations are dissipative; the divergence \( \Delta = \partial p_1/\partial p_1 + \partial p_2/\partial p_2 + \partial q_1/\partial q_1 + \partial q_2/\partial q_2 = -4\alpha \), so an element of volume in phase space contracts like \( e^{-4\alpha \tau} \), and the motion tends to an asymptotic state: an attractor. In his 1962 paper [6], Miles found the fixed-point solutions, both planar \((p_2 = q_2 = 0)\) and non-planar, and performed a stability analysis, the principal finding of which is that if \( \alpha < 0.441 \) there is a finite interval in \( \nu \) bounded by \( \nu_2 = 0.14 - 1.153\alpha^2 + O(\alpha^4) \) and \( \nu_3 = -0.945 + 0.794\alpha^2 + O(\alpha^4) \) for which there are no stable fixed-point solutions. Miles returned to the problem in 1984 [7,9] this time equipped with the tools made available by nonlinear dynamics, and found that within this region of parameter space numerical simulations of his equations show zones of chaotic behaviour characterized by a strange attractor.

Figure 1 displays a plan view of the trajectory of the pendulum bob obtained by integrating the Miles equations over a period of \( 10\tau \); \( \tau \) being the slow time utilized in the

![Figure 1](image_url)

Figure 1. Real space plot showing the trajectory of the pendulum bob given by the Miles equations over a period of \( 10\tau \) during which it reverses its direction of motion. The parameters are \( \alpha = 0.1, \nu = -0.17 \) and \( \omega = 25 \).
theoretical analysis. During this time the pendulum can be seen gradually to change its orbit. It begins swinging in a large quasi-ellipse which has its major axis running from top right to bottom left. This quasi-ellipse precesses clockwise and at the same time becomes smaller and narrows down to a line, after which time it reverses the direction to become a quasi-ellipse precessing anticlockwise. This is an example of a reversal, and the most prominent feature of the behaviour of the forced spherical pendulum. But, although this figure displays the pendulum motion in real space – and is quite hypnotic to watch when plotted continuously on a computer screen – it gives little insight into the structure of the motion, which can only be appreciated over a longer timescale of terms of $\tau$. From here on, we interest ourselves in the behaviour of the pendulum in the four-dimensional phase space $(p_1, q_1, p_2, q_2)$ in the slow time $\tau$.

3. Laboratory experiments and their comparison with numerical simulations

Shortly after Miles had demonstrated the chaotic nature of the Miles equations, a forced spherical pendulum was constructed by one of us (D.J. Tritton), initially as a lecture demonstration of chaos [10]. There were several motives for this. In the first place, the most notable manifestations of chaos in the pendulum are the irregular reversals of its direction of motion. To a geophysicist, these recall the irregular reversals of the Earth’s magnetic field that have been revealed through the paleomagnetic record [11]. Although simple physically motivated models such as the Rikitake equations for a chaotic self-excited dynamo have been proposed to explain the behaviour of the Earth’s magnetic field [12], the forced spherical pendulum provides a simple mechanical model showing irregular reversals that can provide qualitative insight into the consequences of chaos for geomagnetism [13,14]. Secondly, the forced spherical pendulum was one of the examples of chaos that D.J. Tritton wished to discuss in the second edition of his fluid dynamics textbook [15]; the Miles’ equations have a fluid-mechanical application to the problem of liquid sloshing in a vibrating tank, which Miles himself and others have investigated [9,16–18] and to the general problem of periodic water waves in two dimensions [19]. Finally, and most relevant here, the forced spherical pendulum provides a simple physical system whose behaviour can be compared with that of a set of differential equations with chaotic solutions – the Miles equations.

We show a diagram of the experimental set-up in Figure 2. The pendulum is a string of length 28 cm attached to a bob of diameter 6.2 cm and mass 45 g. The upper end of the pendulum is fixed to a rod that may be oscillated in simple harmonic motion by a crank; the drive amplitude used in the experiments reported here is 3 mm. The crank is driven by an electric motor through electronic circuitry that controls its frequency to better than one part in three thousand; this is clearly the critical aspect of the experimental setup, as we wish to uncover changes in behaviour under small changes in the forcing frequency. A mirror is set at $45^\circ$ beneath the pendulum so as to project to one side the horizontal projection of the motion of the bob; this facilitates recording with a videocamera.

In order to compare the Miles equations with experiment, we need $\epsilon$, the ratio of the amplitude of the oscillations (3 mm) to the length of the pendulum, for which we measured the effective length from the period of free oscillations to be 30.9 cm, giving $\epsilon = 0.0097$. The other parameter that we must establish is the damping coefficient in our pendulum apparatus, which is approximately $\alpha = 0.1$ [10] (but note that the damping is linear in Miles’ analysis, but nonlinear in the physical setup). Miles’ stability analysis tells us that
for $\alpha = 0.1$ for driving frequencies $\nu$ between $\nu_3 = -0.937$ and $\nu_2 = 0.142$ there are no stable fixed points. It is in this range of driving frequencies spanning the natural frequency of the pendulum that we should expect to encounter chaotic responses. In Table 1 we present a summary of the results of experiments that scan across this zone of driving frequencies, based on a visual analysis of the asymptotic state of the pendulum [10]; as expected, chaos is present. We may compare this with the numerical solutions of the Miles equations by performing a similar scan of $\nu$ for $\alpha = 0.1$ with a computer, as shown in Table 2. While there is a close correspondence between the two tables, the exact $\nu$ values differ. There are two relevant points to consider, as Tritton and Groves discussed [20], first, that the assignment of the damping coefficient $\alpha = 0.1$ to the laboratory pendulum is only approximate, especially given its nonlinear damping, and second, that at the upper end of the $\nu$ range, where the greater difference is noted between experimental and numerical values, there are found to be coexisting attractors, which complicates the analysis. However, beyond these differences in the details, a comparison of the two tables reveals that the broad structure of the parameter space is the same for the experimental
and model pendulums: in both cases there are three chaotic regions interspersed with periodic windows.

Many aspects of the dynamics are rather easy to discern in numerical simulations, as was discussed extensively by Miles [7]; on the computer it is simple to distinguish between steady, periodic and chaotic behaviour. In the experimental setup such aspects of the dynamics that occur over the slow time $\tau$ are far more difficult to quantify, and analysis of the experimental pendulum behaviour required making painstaking observations over a period of minutes to obtain each datum. The irregular reversals of its direction of precession are the most prominent manifestation of chaos in the laboratory pendulum, as well as being analogous to the reversals of the Earth’s magnetic field, and are most readily accessible to precise determination. Thus, the quantitative results that we extracted from the laboratory pendulum are in the form of histograms showing the frequency of reversals of the precession of the pendulum’s orbit. In order to make quantitative comparisons between numerical integrations of the Miles equations and the experiments, we wished to compile similar histograms of reversals from the Miles equations. The angular momentum $M$ rather than precession is the more fundamental quantity in the theory; and as we have noted above, numerical integrations of angular momentum against time show similar characteristic reversals to those visible in the precession of the pendulum bob. These correspond in real space to the quasi-ellipse of the orbit closing down to a line and subsequently opening out to a quasi-ellipse that is traversed in the opposite sense. However, it is not obvious that in the forced spherical pendulum such an angular-momentum reversal should necessarily coincide with a change in the direction of precession of the axes of the quasi-ellipse. Precession and angular momentum are inextricably linked in the unforced spherical pendulum: the precession of the orbit is in the same sense as the rotation of the bob [1–3]. We must ask ourselves what is the nature of their relationship upon the addition of forcing.

We wish to derive an expression for the precession of the forced spherical pendulum. For the forced spherical pendulum, if $p_1, q_1, p_2,$ and $q_2$ vary slowly, Equation (1) approximates to an ellipse. Say the major axis of the ellipse is at an angle $\phi$ to the original $x$ axis, then we may write it as

$$
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix},
$$

(5)
We must compare this to the parametric equations of an ellipse \( \frac{x^2}{\varepsilon^2 l^2 A^2} + \frac{y^2}{\varepsilon^2 l^2 B^2} = 1 \):

\[
\begin{align*}
\dot{x} &= e^{1/3} l A \cos(\theta - \psi), \\
\dot{y} &= e^{1/3} l B \sin(\theta - \psi).
\end{align*}
\] (6)

After some manipulation, we obtain

\[
\tan 2\phi = \frac{2(p_1 p_2 + q_1 q_2)}{p_1^2 + q_1^2 - p_2^2 - q_2^2} = \frac{2C}{D}
\] (7)

where \( C = p_1 p_2 + q_1 q_2 \) and \( D = p_1^2 + q_1^2 - p_2^2 - q_2^2 \). We are seeking the precession \( d\phi/d\tau \), which is then

\[
\frac{d\phi}{d\tau} = \frac{D \dot{C} - C \dot{D}}{4C^2 + D^2}.
\] (8)

Since

\[
\dot{C} = -2aC + 3MD/4 + q_2
\] (9)

and

\[
\dot{D} = -2aD - 3MC + 2q_1,
\] (10)

the precession may be written as

\[
\frac{d\phi}{d\tau} = \frac{3M}{4} + \frac{q_2 D - 2q_1 C}{4C^2 + D^2}.
\] (11)

So the precession is indeed related to the angular momentum, but has an extra term \((q_2 D - 2q_1 C)/(4C^2 + D^2)\), which originates from the forcing. We show in Figure 3 numerical integrations for various different parameter values showing both the angular momentum and the precession. In general, the forcing term is small, so that an angular-momentum reversal occurs close to a precession reversal, however, this is just a rule of thumb, and it is possible for one quantity to change sign and the other not, so an angular-momentum reversal may occur without a precession reversal, or vice versa. Examples of both types of exception to the rule of thumb are seen in Figure 3: for instance, a precession reversal without an angular-momentum reversal takes place at time 155 in Figure 3(a), while an angular-momentum reversal occurs without an accompanying precession reversal at time 110 in Figure 3(d).

In early work with the pendulum apparatus, it was thought that precession and angular momentum reversed as one; it was affirmed that ‘precession of the opposite sense to the circulation has never been seen’ [10]. This, we can now see, is not true, albeit precession and angular momentum reversals in general do occur close together, and it is not at all easy to distinguish exceptional cases in the experimental setup. It is important to bear this in mind with regards to the experimental results, as the procedure by which the data were obtained was to note ‘the times during which the sense of motion remained the same. (One stopwatch was stopped and another started each time circulation and/or precession of a new sense became perceptible)’ [10]. To compare with the experimental reversal histograms, shown in Figure 4, we compiled histograms of both angular momentum \( M \)
reversals and precession $d\phi/d\tau$ reversals from numerical integrations of the Miles equations.

The sequence of histograms in Figure 4 displays experimental histograms obtained from the laboratory pendulum for four values of the driving frequency together with the statistics of angular momentum $M$ reversals and precession $d\phi/d\tau$ reversals from the Miles equations for the same four values of $\nu$ within the chaotic range. The damping is set
to $\alpha = 0.1$ for the numerical solutions to coincide with the experiments, although it must always be borne in mind that the correspondence is not totally clear given the nonlinear damping in the experiment. Despite this, in the first place, it is clear that there is a good correspondence between experimental and numerical results. They show the same basic features of a block of reversals at short intervals, plus isolated reversals further out. The largest peak – the modal reversal interval – compares well between the experimental and numerical results. It is notable that at some of the $\nu$ values the plots show bimodality;
the presence of two peaks of the distribution. This shows the presence of multiple time scales in the dynamics that we may also discern in the numerical angular momentum and precession plots of Figure 3.

A comparison of the numerical results of angular-momentum and precession reversal intervals shows that their statistics, while similar, are not identical. This begs the question of whether the observers taking the statistics of the experimental pendulum (physics students at Newcastle) were paying attention to the precession or the circulation, or at times one and at other times the other. If we compare each of the four sets of data, we may note that the experimental data coincide better with the numerical data from angular momentum reversals for $\alpha = 0.1$ for (from left to right) (a) $v = -0.17$, and (b) $v = -0.52$. Available in colour online.

Figure 4. Experimental histograms (top; green) showing the fraction of reversals against the period between successive reversals, and numerical precession $d\phi/d\tau$ (centre; blue) and angular momentum $M$ (bottom; red) reversal histograms for $\alpha = 0.1$ for (from left to right) (a) $v = -0.17$, and (b) $v = -0.52$. Available in colour online.
other parameter values precession reversals in which the quasi-ellipse changes its sense of precession are more obvious.

One concern was whether varying the tolerance of the numerical integration algorithm would affect the statistical results. We show angular momentum against time and angular-momentum reversal histograms for $\alpha = 0.1$, $\nu = 0.0$ for tolerance $10^{-3}$, $10^{-5}$ and $10^{-7}$ in Figure 5. The sequence of plots of angular momentum $M$ against time shows the exponential separation of trajectories that is a hallmark of chaos. The starting parameters were identical except for the tolerance used in the Runge–Kutta integration algorithm. One can see that the more accurately the integration is performed, the longer the solution remains close to the ultimately unobtainable absolute solution. Any solution with finite accuracy diverges from this, and in a short time becomes totally uncorrelated. A trajectory calculated numerically is not a true chaotic orbit, but corresponds to a set of short lengths of true orbits, starting from different initial conditions. A tenfold increase in the tolerance maintains the solution close to the absolute solution for about $10\tau$ longer, so it would be impossible in practice to find the absolute solution for more than the first couple of hundred $\tau$ here. Since we have used a tolerance of $10^{-4}$ in general, all the solutions shown are entirely unrelated to the initial conditions used – $(p_1, p_2, q_1, q_2) = (0, 0, 0, 1)$ – from

Figure 4. (From left to right) (c) $\nu = -0.70$, and (d) $\nu = -0.91$. Available in colour online.
about 50τ from the start. In the reversal histograms, sampled from 500τ to 5500τ from the start, the solutions had long since diverged from the absolute solution by the time sampling commenced. For this reason the histograms are different in detail; the detailed structure of the trajectories is quite different. The significant point is that the general features remain the same; the peak is in the same place – at 7τ – and there are similar patterns of isolated reversals occurring at longer intervals. This is what we should expect from the numerical integration correctly shadowing true trajectories on the chaotic attractor [21]. Thus, we may be confident in comparing the numerical results with those from the experiment.

It is also interesting to note that at the υ value, υ = 0.0, used in Figure 5 the reversal histograms display trimodality; there are three time scales in the dynamics compared with the one or two in the histograms of Figure 4. In Figure 6 we plot the modal and mean angular momentum $M$ and precession $\text{d}\phi/\text{d}r$ reversal intervals for $\alpha = 0.1$ in the $\nu$ range $(-1, 0.15)$. Because of the presence of more than one peak – multimodality – at many values of $\nu$, the largest peak, that is to say the mode, of the distribution, may be quite different.

Figure 5. Numerical simulations of angular momentum $M$ plotted against time (left) and angular momentum reversal histograms (right) for $\alpha = 0.1$, $\nu = 0.0$ for (from top to bottom) numerical integration local error (a) $10^{-3}$, (b) $10^{-5}$, and (c) $10^{-7}$. Available in colour online.
from the mean. At the top end of the $v$ range there are no reversals occurring; the pendulum is executing a nonplanar fixed-point orbit. No reversals occur at first when $v$ drops below $v_2$. They first appear when $v = 0.07$, when they have a mean period of $\sim 6\tau$, coinciding with the analysis of the Hopf bifurcation that has occurred. There are no reversals for the single case $v = 0.04$, but from there on the mean angular momentum reversal interval rises to a peak at $v = 0$, when it declines again. For the rest of the $v$ range the mean angular momentum reversal interval decreases slowly, although the graph shows many oscillations about its mean level. The mean precession reversal interval shows somewhat different behaviour, having a generally lower mean, showing that precession reversals without accompanying angular momentum reversals are a common feature. The largest periodic window in the $v$ range occurs from $-0.3245$ to $-0.37$. In this region of the mean reversal interval graphs, one sees a short smooth section, in contrast with the noisier behaviour elsewhere; chaos manifests itself as noisy intervals in contrast to periodic behaviour that gives rise to smooth intervals. Although below $v = -0.93$ there is no chaos but steady planar motion, reversals are seen on the mean reversal interval graphs as the transient solution has small-amplitude oscillations about $M = 0$. As for the modal reversal interval graphs, they reflect how the dominant periodicity varies with $v$, it being particularly clear that short $3\tau$ reversal intervals predominate for the bottom end of the $v$ range.
4. Discussion

Our comparison of the statistics of reversal intervals in the experimental pendulum with the numerical integration of the Miles equations reveals rather good agreement between the two. Not all numerical investigations of the forced spherical pendulum have used the Miles equations; there have been several pieces of research that forego the approximations these equations imply [22–24]. Tritton and Groves [20] compared the Lyapunov exponents they calculated using the Miles equations with those Bryant [24] obtained integrating the full equations and found very good concordance. For our purposes here the Miles equations display very clearly how the trajectory of the pendulum evolves in slow time; the full equations obscure this insight into the dynamics. Moreover, the Miles equations have the advantage of permitting more rapid computations governed by just two parameters in which the long-term behaviour of the pendulum is patent. The good agreement we find between experiment and simulations demonstrates that the approximations introduced in the Miles equations do not of significantly alter the behaviour of the system for our purposes here of obtaining good statistics of the dynamics.

We have an important instance in the case of the Earth’s magnetic field where a sequence of reversals recorded in magnetic rocks is the only experimental evidence we have to understand the dynamical system, which makes the reversals of the spherical pendulum a useful testbed, as here we do have access to the mathematics of the dynamical system itself.

Whereas in the unforced spherical pendulum angular momentum and precession have an unvarying relationship, this association is looser in the forced case. We have shown that there is an addition forcing term in the expression for the precession, so that although the two quantities in general go hand in hand, it is possible for the angular momentum to reverse – the pendulum’s quasi-ellipse narrows down to a line to emerge as a quasi-ellipse with the opposite circulation – without the precession doing so, or, more commonly, for the precession to reverse while the angular momentum does not; the quasi-ellipse alters its sense of precession while the circulation remains the same.

Of course, the experimental pendulum inevitably differs in some respects from the theoretical pendulum of the Miles equations, principally because in the latter the damping is linear, whereas in the real version it is not. Despite this, the theoretical model has proven remarkably close in its behaviour to the laboratory version of Tritton [10]. Other experimental investigations of the forced spherical pendulum have been reported [25,26] – whose results were compared with those of Tritton [10] by Tritton and Groves [20] – and newer technology offers many as yet unexplored possibilities for obtaining quantitative data from a pendulum setup in a more automated way than was possible with ours. We hope that others may be enticed into the field and encourage them to follow Robert Hooke [5] and ourselves and to build a spherical pendulum.

Acknowledgements

Julian Cartwright wishes to dedicate this article to the memory of his co-author David Tritton, who would have been 75 this year: It is based on work performed by me in 1988, for my undergraduate thesis in theoretical physics at the University of Newcastle upon Tyne under David’s supervision [27]. David referred to these results in work he subsequently published on the spherical pendulum [14,20,28]. We had begun to prepare this article, and in 1998 we were planning that he should visit me, during which time, among other things, undoubtedly we would have finally have
got around to finishing writing it up. However, that never happened as he died unexpectedly on
24 April 1998 during a stay in the USA [29]. This article, then, both makes these findings available
at last, and provides a small memorial to David as well as to physics at Newcastle upon Tyne;
the School of Physics where David worked and I was an undergraduate student has been
closed, it having been decided by the powers that a physics department is not of interest to the
university.

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